Efficient Calculation of the Green's Function for the Rectangular Cavity

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***Abstract*—An efficient method is developed for the rapid and accurate calculation of the potential Green's function in the rectangular cavity. The proposed method is based on the Ewald sum technique and converts the slowly convergent cavity Green's function into the sum of two exponentially convergent series. The transformed series achieves fast convergence with only a small number of terms in the calculation.**

***Index Terms*— Green's function, rectangular cavity, Ewald sum.**

I. Introduction

In the moment method analysis of the microwave structures involving rectangular cavities, the fast and accurate calculation of the Green's function is very important [1], [2]. The frequently used modal form of the cavity Green's function is quite slowly convergent, particularly when the observation point is close to the source point. Therefore, it is often impractical to calculate the triply infinite Green's function series directly [1].

A hybrid ray-mode calculation method of the Green's function has been proposed in [1] for electrically large cavities. In [2], an efficient calculation method has been proposed for the potential Green's function arising from the periodic source distribution inside the rectangular waveguide, which can be easily applied to the cavity Green's function. The Green's function has been represented by the first few terms of the modal expansion plus quasi-static correction terms, the latter of which have been converted into the rapidly convergent form.

This letter proposes another method for the rapid and accurate calculation of the rectangular cavity Green's function based on the Ewald sum technique [3], [4]. The Ewald sum technique has been derived and used for the effective calculation of the potential function due to the two-dimensional [4] and three-dimensional [3] periodic source distributions in homogeneous media and also proved to be effective in the calculation of the rectangular waveguide Green's function [5]. For the rectangular cavity problems, the application of Ewald sum technique expresses the Green's function as the hybrid sum of the modal and the image series. Both of the two series are rapidly convergent, and therefore, the Green's function can be calculated accurately with only a small number of terms in the series.

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II. Theory

The vector potential Green's function for the rectangular cavity can be represented by the following dyadic form:

$$\overline{\overline{G}}_A = \hat{x}\hat{x}G_{Axx} + \hat{y}\hat{y}G_{Ayy} + \hat{z}\hat{z}G_{Azz} \tag{1}$$

$$\overline{\overline{G}}_F = \hat{x}\hat{x}G_{Fxx} + \hat{y}\hat{y}G_{Fyy} + \hat{z}\hat{z}G_{Fzz} \tag{2}$$

where the subscript A and F designates the magnetic and the electric vector potential, respectively. Since the proposed method can be applied to all components of the Green's function in a similar way, only the G_{Axx} component will be presented here for brevity.

Each component of the dyadic Green's function can be expressed in two different forms. One is the spectral representation in terms of modal functions of the cavity [see (3)], and the other is the spatial expansion in terms of images produced by the cavity walls [see (4)].

$$G_{Axx} = \frac{\mu}{abc}\sum_{m,\,n,\,p=0}^{\infty}\frac{\varepsilon_m\varepsilon_n\varepsilon_p}{\alpha_{mnp}^2}\cos\frac{m\pi x}{a}\cos\frac{\mathrm{m}\pi\mathrm{x}'}{\mathrm{a}} \cdot\sin\frac{\mathrm{n}\pi\mathrm{y}}{\mathrm{b}}\sin\frac{\mathrm{n}\pi\mathrm{y}'}{\mathrm{b}}\sin\frac{\mathrm{p}\pi\mathrm{z}}{\mathrm{c}}\sin\frac{\mathrm{p}\pi\mathrm{z}'}{\mathrm{c}}$$

$$\varepsilon_i = \begin{cases}1, & i=0\\ 2, & i\neq 0\end{cases}$$

$$\alpha_{mnp}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2 - k^2 \tag{3}$$

$$G_{Axx} = \frac{\mu}{4\pi}\sum_{m,\,n,\,p=-\infty}^{\infty}\sum_{i=0}^{7}A_i^{xx}\frac{e^{-jkR_{i,\,mnp}}}{R_{i,\,mnp}}$$

$$A_i^{xx} = \begin{cases}+1, & o=0,\,3,\,4,\,7\\ -1, & i=1,\,2,\,5,\,6\end{cases}$$

$$R_{i,\,mnp} = \sqrt{(X_i+2ma)^2 + (Y_i+2nb)^2 + (Z_i+2pc)^2}$$

$$X_i = \begin{cases}x-x', & i=0,\,1,\,2,\,3\\ x+x', & i=4,\,5,\,6,\,7\end{cases},\quad Y_i = \begin{cases}y-y', & i=0,\,1,\,4,\,5\\ y+y', & i=2,\,3,\,6,\,7\end{cases},$$

$$Z_i = \begin{cases}z-z', & i=0,\,2,\,4,\,6\\ z+z', & i=1,\,3,\,5,\,7\end{cases} \tag{4}$$

where a, b, and c are the dimensions of the cavity.

The image expansion of the Green's function can be divided into the following two series according to the identity derived by Ewald [3], [4]:

$$G_{Axx} = G_{Axx1} + G_{Axx2} \tag{5}$$

$$G_{Axx1} = \frac{\mu}{4\pi} \sum_{m,\,n,\,p=-\infty}^{\infty} \sum_{i=0}^{7} A_i^{xx} \frac{2}{\sqrt{\pi}} \int_0^E e^{-R_{mnp}^2 s^2 + \frac{k^2}{4s^2}}\, ds \tag{5a}$$

$$G_{Axx1} = \frac{\mu}{4\pi} \sum_{m,\,n,\,p=-\infty}^{\infty} \sum_{i=0}^{7} A_i^{xx} \frac{2}{\sqrt{\pi}} \int_E^{\infty} e^{-R_{mnp}^2 s^2 + \frac{k^2}{4s^2}}\, ds \tag{5b}$$

where E is an adjustable parameter in the Ewald sum method. The G_{Axx1} series can be converted into the following form:

$$G_{Axx1} = \frac{\mu}{16abc} \sum_{m,\,n,\,p=-\infty}^{\infty} \sum_{i=0}^{7} A_i^{xx} \int_0^E e^{-\frac{\alpha_{mnp}^2}{4s^2}} \frac{1}{s^3}\, ds \cdot e^{j\left[\frac{m\pi}{a} X_i + \frac{n\pi}{b} Y_i + \frac{p\pi}{c} Z_i\right]}. \tag{6}$$

After the closed-form evaluation of the integral [3], the above series can be simplified as follows:

$$G_{Axx1} = \frac{\mu}{abc} \sum_{m,\,n,\,p=0}^{\infty} \frac{\varepsilon_m \varepsilon_n \varepsilon_p}{\alpha_{mnp}^2} e^{-\frac{\alpha_{mnp}^2}{4E^2}} \cos\frac{m\pi x}{a} \cos\frac{m\pi x'}{a} \cdot \sin\frac{n\pi y}{b} \sin\frac{n\pi y'}{b} \sin\frac{p\pi z}{c} \sin\frac{p\pi z'}{c}. \tag{7}$$

The integral in G_{Axx2} series can also be evaluated in closed-form [3], [4], and the resultant series can be simplified as follows:

$$G_{Axx2} = \frac{\mu}{4\pi} \sum_{m,\,n,\,p=-\infty}^{\infty} \sum_{i=0}^{7} A_i^{xx} \cdot \frac{\mathrm{Re}[e^{-jkR_{i,\,mnp}} \mathrm{erfc}(R_{i,\,mnp} E - jk/2E)]}{R_{i,\,mnp}} \tag{8}$$

where $\mathrm{Re}[A]$ designates the real part of a complex number A. The G_{Axx1} and G_{Axx2} series can be interpreted as the exponential/error-function weighted modal/image expansion of the Green's function, respectively. Clearly, the G_{Axx1} series is exponentially convergent, and the G_{Axx2} series is also very rapidly convergent due to the presence of the complementary error function $[\mathrm{erfc}(x)]$ which decays at the rate of e^{-x^2}/x. Therefore, only a small number of terms are required to obtain a convergent result by the proposed method.

III. Numerical Result

The proposed method is tested on a square cavity whose side dimensions $(a = b = c = L)$ are all 0.99 λ_0, where λ_0 is the free-space wavelength.

Fig. 1 compares the convergence of the Green's function using the modal expansion [see (3)] and the Ewald sum method for three different cases. The source point is fixed at the center of the cavity $(x' = y' = z' = 0.5L)$ and three observation points are selected along the diagonal of the cavity, far from the source point $(x = y = z = 0.1L)$, near the source point $(x = y = z = 0.49L)$, and around the midpoint of these two $(x = y = z = 0.3L)$. In this case, the three components of the vector potential are the same. The reference values in the calculation of the relative errors are obtained by the Ewald sum method with sufficiently large number of terms included in the calculation.

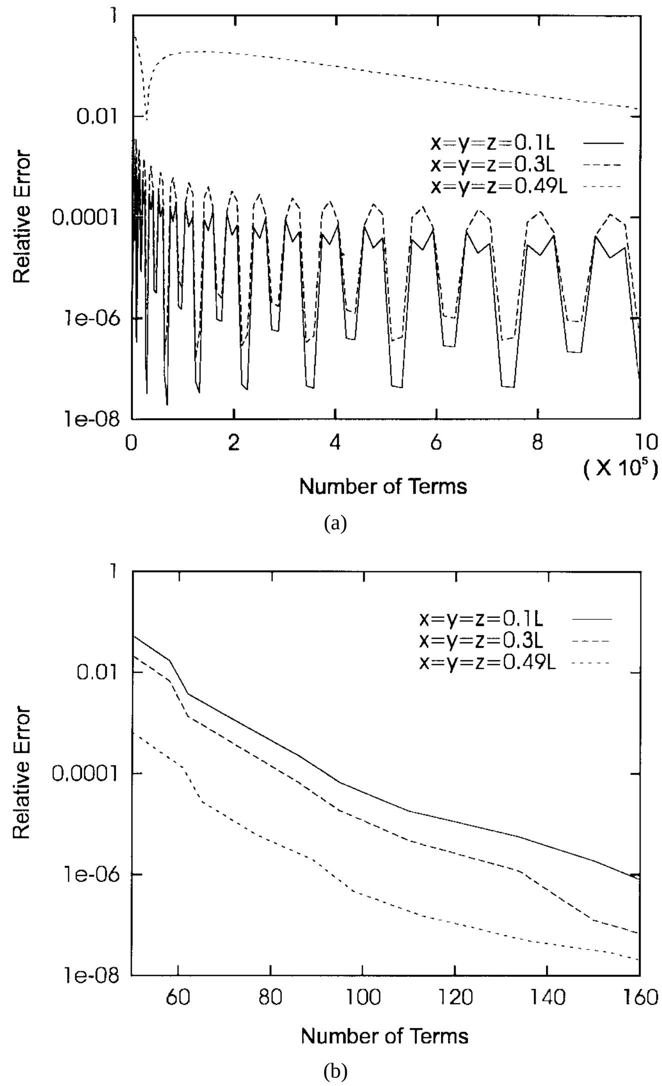

Fig. 1. Typical convergence behavior of the potential Green's function ($G_{Axx} = G_{Ayy} = G_{Azz}$: $x' = y' = z' = 0.5L$). (a) Modal expansion. (b) Ewald sum method.

In general, the convergence of the modal Green's function [Fig. 1(a)] gets worse as the observation points become nearer to the source point. The slow and/or oscillating convergence of this method limits the accuracy obtainable with this method even though quite a number of terms are included in the calculation. In contrast, the calculations by the Ewald sum method show very rapid convergence in all three cases considered [Fig. 1(b)].

In applying the Ewald sum technique, the parameter E should be predetermined. Larger value of E accelerates the convergence of the spatial (G_{Axx2}) series at the expense of slower convergence of the spectral (G_{Axx1}) series and vice versa. Since the spatial series requires more calculation than the spectral series for one term of the series, it is better to

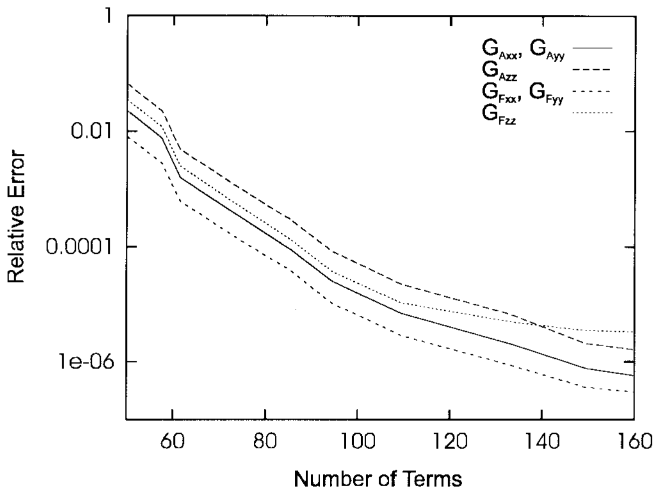

Fig. 2. Average convergence behavior of the Ewald sum method ($z = z' = 0.5L$).

TABLE I
AVERAGE RELATIVE ERROR FOR VARIOUS SOURCE POINTS IN THE CALCULATION OF THE POTENTIAL GREEN'S FUNCTION BY THE EWALD SUM METHOD ($z = z' = 0.5L$)

($\times 10^{5}$)

x', y'	0.05a	0.15a	0.25a	0.35a	0.45a
0.05b	2.18	1.81	6.49	2.38	1.93
0.15b	1.81	0.87	1.03	2.77	1.29
0.25b	6.49	1.03	2.02	1.95	0.92
0.35b	2.38	2.77	1.95	0.94	1.23
0.45b	1.93	1.29	0.92	1.23	2.28

choose the E value which requires less terms in the calculation of the spatial series than in the spectral series. For this purpose, the E value has been chosen to achieve enough convergence in the spatial series with images within the diagonal distance of the cavity from the observation point. This took around 20 terms for the convergence of the spatial series.

Fig. 2 shows the average convergence behavior of the Ewald sum technique for various components of the vector potential Green's function. The data shown in Fig. 2 have been obtained through averaging Green's function calculation results over 625 different pairs of source and observation points on the $z = z' = 0.5L$ plane. On average, it takes only about 90 and 110 terms to achieve 10^{-4} and 10^{-5} convergence for all components of the Green's function, respectively. Therefore, about 100 term calculations seem sufficient by the proposed method for use with the numerical analysis of microwave structures involving the rectangular cavities. Table I shows the relative error for various source points, when about 100 terms on average are included in the calculation by the Ewald sum method. Each data in the table represents calculation results averaged over all components of the vector potential Green's function (electric and magnetic) for 25 different observation points on the $z = z' = 0.5L$ plane. In this case, the maximum error is 6.49 $\times 10^{-5}$ and the total average error is 2.07 $\times$ 10^{-5}.

IV. CONCLUSION

This letter proposed an efficient and accurate method based on the Ewald sum technique to calculate the potential Green's function in the rectangular cavity. This method can drastically improve the convergence rate of the Green's function series and can be used beneficially in the moment method analysis of various microwave structures involving rectangular cavities.

REFERENCES

[1] D. I. Wu and D. C. Chang, "A hybrid representation of the Green's function in an overmoded rectangular cavity," *IEEE Trans. Antennas Propagat.*, vol. 36, pp. 1334–1342, Sept. 1988.

[2] W. F. Richards, K. Mcinturiff, and P. S. Simon, "An efficient technique for computing the potential Green's functions for a thin, periodically excited parallel plate waveguide bounded by electric and magnetic walls," *IEEE Trans. Microwave Theory Tech.*, vol. 35, pp. 276–281, Mar. 1987.

[3] P. P. Ewald, "Die Berechnung optischer und elektrostatischen Gitterpotentiale," *Ann. Phys.*, vol. 64, pp. 253–268, 1921.

[4] K. E. Jordan, G. R. Richter, and P. Sheng, "An efficient numerical evaluation of the Green's function for the Helmholtz operator on periodic structures," *J. Comput. Phys.*, vol. 63, pp. 222– 235, 1986.

[5] M.-J. Park and S. Nam, "Efficient calculation of the Green's function in rectangular waveguides," in *IEEE Antennas Propagat. Int. Symp. Dig.*, vol. 4, July 1997, pp. 2354–2357.